The Case for Incomplete Markets*

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Abstract

We propose a new welfare criterion that allows us to rank different financial market structures in the presence of belief heterogeneity. We analyze economies with complete and incomplete financial markets and/or restricted trading possibilities in the form of borrowing limits or transaction costs. We describe circumstances under which some restrictions on financial markets are desirable according to our welfare criterion.

Keywords: social welfare, heterogeneous beliefs, incomplete markets, financial regulation

1 Introduction

The conventional wisdom in the economics profession is that complete markets are a good thing. The welfare theorems state that market outcomes

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are Pareto optimal, that any optimal allocation can be realized by trade in complete markets with an appropriate lump-sum transfer scheme, and that putting limits on trade, by foreclosing trading opportunities, leaves potential mutual gains unrealized. This “wisdom” has practical consequences. Arguments for the privatization of social security and against the deregulation of financial markets rely on the assertion that barriers to trade are bad things. “Free trade” is the benchmark environment against which all international trade regimes are measured.

Complete markets have their critics. Some traders have market power, whose exploitation can only be limited by constraining trade. Market outcomes, though optimal, are bad in other ways; since lump-sum transfers are impossible, the sacrifice of dead-weight loss is necessary to achieve other goals. These critiques are empirical. The degree of market power could be large or small. Lump-sum transfers are not so much impossible as they are difficult to execute. Consequently, these concerns are considered to be second-order.\footnote{Arnold Harberger (1954) even estimates that the dead-weight welfare loss due to monopoly is on the order of one tenth of one percent of GDP. Bergson (1973), with equally extreme assumptions, can get a number 100 times as large.}

We offer here a more fundamental critique: When markets allocate contingent claims among expected-utility-maximizing agents, Pareto optimality with \textit{ex ante} beliefs is an inappropriate welfare criterion except in the negligible instance where all traders have identical beliefs over states of the world. This critique is detailed in section 3, after an infinite-horizon model of trade in a single consumption good with complete markets is developed in section 2. If the “true distribution of states” was known to an omnipotent social planner, Pareto calculations with correct beliefs is an obvious fix. Omnipotent social planners are rare, however, and without them there is no alternative welfare requirement that obviously ameliorates the issues raised in section 3. We investigate the magnitude of the problem through simulations. The simulations of section 4 examine several policy alternatives to complete markets in Markovian instances of the model developed in the next section, and explore the size and location of the set of potentially true distributions for which the policies would lead to a true welfare improvement with respect to several distinct welfare criteria. We conclude in section 5 with a discussion of the theoretical and the policy implications of our findings.
2 The model

We assume that time is discrete and begins at date 0. At each date a state is drawn from the set \( S = \{1, \ldots, S\} \). The set of all sequences of states is \( \Sigma \) with representative sequence \( \sigma = (s_0, s_1, \ldots) \), called a path. Let \( \sigma^t = (s_0, \ldots, s_t) \) denote the partial history through date \( t \). We use \( \hat{\sigma} | \sigma^t \) to indicate that a path \( \hat{\sigma} \) coincides with a path \( \sigma \) up through period \( t \).

The set \( \Sigma \) together with its product sigma-field is the measurable space on which everything is built. Let \( P^0 \) denote the “true” probability measure on \( \Sigma \). For any probability measure \( P \) on \( \Sigma \), \( P_t(\sigma) \) is the (marginal) probability of the partial history \( \sigma^t \): \( P_t(\sigma) = P(\{\sigma^t\} \times S \times S \times \cdots) \).

In the next few paragraphs we introduce a number of random variables of the form \( x_t(\sigma) \). All such random variables are assumed to be date-measurable; that is, their value depends only on the realization of states through date \( t \). Formally, \( F_t \) is the \( \sigma \)-field of events measurable at date \( t \), and each \( x_t(\sigma) \) is assumed to be \( F_t \)-measurable.

An economy contains \( I \) consumers, each with consumption set \( \mathbb{R}_+ \). A consumption plan \( c : \Sigma \to \prod_{t=0}^{\infty} \mathbb{R}_+ \) is a sequence of \( \mathbb{R}_+ \)-valued functions \( \{c_t(\sigma)\}_{t=0}^{\infty} \) in which each \( c_t \) is \( F_t \)-measurable. Each consumer is endowed with a particular consumption plan, called the endowment stream. Consumer \( i \)'s endowment stream is denoted \( e_i \). The aggregate endowment stream is denoted by \( \bar{e} \):

\[
\bar{e}_t(\sigma) = \sum_{i=1}^{I} e_i^t(\sigma).
\]

An allocation is a profile of consumption plans, one for each individual. The allocation \( (c^1, \ldots, c^I) \) is feasible if for all \( \sigma \) and \( t \), \( \sum_i c_i^t(\sigma) - e_i^t(\sigma) = 0 \).

Consumer \( i \)'s preferences on consumption plans are described by a belief or forecast distribution \( P^i \), a probability distribution on \( \Sigma \), a discount factor \( 0 < \beta_i < 1 \), and a payoff function \( u_i : \mathbb{R}_+ \to \mathbb{R} \). The utility consumer \( i \) assigns to consumption plan \( c \) is the expectation of the average discounted value of the sequence of payoff realizations:

\[
U_{P^i}(c) = (1 - \beta_i) E_{P^i} \left\{ \sum_{t=0}^{\infty} \beta_t^i u_i(c_t(\sigma)) \right\}.
\]

(1)

Notice that beliefs are indexed by individual names. Different individuals may believe different things about the future, and these beliefs need not
coincide with what will actually happen. The true state process is a stochastic process on \( S \), characterized by a probability distribution \( P^0 \) on \( \Sigma \), and it may be the case that for no distinct \( i, j \geq 0 \) does \( P^i = P^j \). We will impose some constraints on how different beliefs can be.

We assume the following properties of the payoff function:

**A1.** Each \( u_i : \mathbb{R}_{++} \to (-\infty, \infty) \) is \( C^1 \), strictly increasing and strictly concave.

**A2.** Each \( u_i \) satisfies an Inada condition at 0: \( \lim_{c \downarrow 0} u'_i(c) = \infty \).

We assume the following properties of the aggregate endowment:

**A3.** The aggregate endowment is uniformly bounded from above and away from 0:

\[ \infty > F = \sup_{t, \sigma} \bar{e}_t(\sigma) \geq \inf_{t, \sigma} \bar{e}_t(\sigma) = f > 0. \]

Finally, we assume that anything is possible at any date, and that individuals believe this to be true:

**A4.** For all individuals \( i \), all dates \( t \) and all paths \( \sigma \), the distributions \( P^i_t(s_t | \sigma^{t-1}) \) for \( i \geq 0 \) have full support.

While our analytical results are proven at this level of generality, our simulations will be undertaken in the simpler domain wherein all distributions are homogeneous Markov processes. The beliefs of individual \( i \) will be described by an initial distribution \( \pi_i \) on \( S \) and a Markov transition matrix \( \Pi_i = \{ \pi_{jk}^i \} \). Then:

\[ P^i_t(\sigma | \sigma_0) = \prod_{\tau=1}^{t} \pi^i(\sigma_\tau | \sigma_{\tau-1}). \]  

(2)

We call so-constructed belief system *dogmatic* because a type-\( i \) agent never updates his forecast distributions. We prohibit any learning. We will discuss below why we believe our principal results are robust to learning behavior.

We measure distance between beliefs \( P^i \) and the true probability \( P^0 \) using relative entropy, or Kullback-Liebler divergence. The relative entropy of belief \( P^i \) is:

\[ KL(P^i | P^0) = \sum_j \pi^0_j \sum_k \ln(\pi^0_{jk} / \pi^i_{jk}), \]  

(3)

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where $\bar{\pi}^0 = \{\bar{\pi}^0_j\}$ is the stationary distribution of a Markov chain with transition matrix $\Pi^0$. It is easy to verify that $KL(P^i|P^0) \geq KL(P^0|P^0) = 0$.

**Definition.** Beliefs of a type-\(i\) agent are more accurate than beliefs of a type-\(j\) agent if

$$KL(P^i|P^0) \leq KL(P^j|P^0).$$

We will often refer to agents as being optimistic or pessimistic. We say that a type-\(i\) agent is optimistic after history $\sigma^t$ if $E^i[e^i|\sigma^t] > E^0[e^i|\sigma^t]$. **Pessimism** is defined analogously.

### 3 The welfare economics of heterogeneous beliefs

The welfare analysis of market outcomes begins with the Pareto order, taking preferences as given. "Tastes," say Stigler and Becker [1977, p. 76], “are the unchallengeable axioms of a man’s behavior: he may properly (usefully) be criticized for inefficiency in satisfying his desires, but the desires themselves are data.” Tastes, they say, “are not capable of being changed by persuasion.”

In contingent-claims markets, “Pareto optimality” is taken to be with respect to *ex ante* preferences; that is, *ex ante*, or time-0, expected utility. While we do certainly agree that tastes for apples and oranges, work and leisure, etc., are to be taken as given, we dispute the claim that *ex ante* preferences on contingent claims are above dispute. Time-separable EU preferences have three components: attitudes towards risk, the rate of time preference, and beliefs about the realization of states. While risk attitudes and discount factors may be unarguable, beliefs are not. When market participants have different beliefs, not all can be right, and those who are not would certainly revise their beliefs if they saw their error, had additional information, and so forth.

#### 3.1 Spurious unanimity

Ithaca NY, the home of three of us, has a pedestrian mall. It is still serviceable, but would benefit from renovation. The work, however, will be costly.

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2The point of our paper is to replace this view with an even stronger claim about tastes from which this follows.
Half the town believes that revitalization will enhance Ithaca’s attraction as a summer tourist destination. Crowds of tourists will bring more business opportunities and badly needed tax revenues. Half the town believes that revitalization will make downtown more pleasant without materially perturbing downtown’s summer population density, thereby enhancing the quality of life. The town is unanimous in its support for the project. Is unanimity of preference a good argument for undertaking the project? Not according to Mongin [2005], who calls this problem “spurious unanimity”. He argues that not only preferences themselves, but the reasons why people hold the preferences they have, need to be considered in making welfare claims. This is clear in the mall-renovation case. Suppose that many editorials have appeared in the local newspaper, many public meetings have been held, and the issue has been thoroughly aired. It is common knowledge, then, that individuals believe different things. It is common knowledge, then, that if the mall is renovated, half the town will be unhappy with the result. It is common knowledge that the renovation cannot be an \textit{ex post} Pareto improvement. There is disagreement only over who is in which half. Suppose there are \( N \) different possible states of the world rather than 2, and that the population is divided equally into \( N \) groups. Individuals in any group will benefit from a proposal only if “their” state of the world occurs and will be harmed otherwise, and each individual is sure that the state beneficial to him will occur. It is then common knowledge that only fraction 1/\( N \) of the population will be made better off, that fraction \( N - 1/N \) will be made worse off. Imagine that \( N \) is large. The justification of the proposal by \textit{ex ante} Pareto optimality is not at all compelling.

The problem of spurious unanimity is even more compelling when expected utility decision makers choose over alternatives with random payoffs. Imagine now that two decision-makers are choosing between two policies, \( A \) and \( B \). Policy \( A \) gives outcome \( a \) on event \( E \) and \( b \) on \( E^c \). Policy \( B \) is the mirror-image; it gives outcome \( b \) on \( E \) and \( a \) on \( E^c \). Individuals 1 and 2 each have a payoff function and a prior belief, which are as follows:

\[
\begin{align*}
\text{Individual 1} & \quad u_1(a) = 1, \quad u_1(b) = 0, \quad \rho_1(E) = 0.99, \\
\text{Individual 2} & \quad u_2(a) = 0, \quad u_2(b) = 1, \quad \rho_2(E) = 0.01.
\end{align*}
\]

Each individual prefers policy \( A \) to policy \( B \). Unanimity is a consequence of their divergent beliefs. Given their payoff functions, if they shared a common belief they could never agree on a policy except in the trivial case where they both believe each state is equally likely.
Savage's (1954) subjective expected utility representation theorem delivers for each preference order satisfying his axioms a payoff function and a probability vector that together generate an additively separable representation over state-contingent payoffs. Although Savage's theorem does not compel any particular interpretation, economists and game theorists typically take the payoff function as representing tastes, such as attitudes towards risk, and the probability distribution as representing beliefs. In the common interpretation, such preferences come with their justifications encoded in the preference order, and so no other information than the preference orders themselves are needed to detect spurious unanimity. This argument, however, is not correct; the interpretation of probabilities as beliefs requires an extra-preference justification.

The argument that one can extract beliefs from preferences depends critically on the uniqueness of the probability distribution in Savage's representation theorem. Unfortunately, this view is not correct. Suppose that a preference order for acts mapping states \( s \in S \) to outcomes \( y \in Y \) has an expected utility representation: a payoff function \( u : Y \to \mathbb{R} \) and a probability distribution \( p \) on \( S \). The uniqueness theorem states that if \( v \) and \( q \) combine to give another expected utility representation of the same preference, then \( v \) is a positive affine transformation of \( u \) and \( q \) equals \( p \). This result, however, is limited to state-independent payoff representations. If we allow that tastes can depend upon states, so that payoff functions can map \( S \times Y \) into \( \mathbb{R} \), then the only thing unique about the probability distribution is its support. For any \( q \) with support identical to \( p \), there is a state-dependent payoff function \( v \) such that \( v \) and \( q \) combine to represent the preference. Savage’s axioms include a “state-independence” assumption, that preference conditional on two distinct non-null states are identical. This allows the possibility of a state-independent payoff function, but it does not rule out state-dependent payoffs. Together with the other axioms, the only requirement it imposes on \( v : S \times Y \to \mathbb{R} \) is that the function \( v(s', \cdot) : Y \to \mathbb{R} \) is a positive affine transformation of \( v(s, \cdot) : Y \to \mathbb{R} \) (whenever \( s \) and \( s' \) are both non-null).

Interpreting probabilities in expected utility representations as likelihood assessments requires uniqueness of the probability distribution in the larger class of state-dependent expected utility representations. Pinning the positive affine transformations down to translations is the necessary condition for deriving uniqueness, requires an extension to the structure of preferences that is not revealed in choice behavior. We conclude that if one insists that individuals preferences have expected utility representations, then the commitment
that individuals have identical beliefs can only be justified by non-choice considerations even when individuals preferences can be represented by (perhaps different) state-independent payoff functions and a common probability distribution.3

3.2 The \textit{ex ante} welfare economics of contingent claims

Because beliefs are not above dispute, we are concerned with two Pareto orders: The usual welfare analysis is concerned with the \textit{ex ante} Pareto order, and because individuals would choose to adopt the true distribution if only they knew it, we are also concerned with the \textit{true} Pareto order which is the order that obtains when each individual computes expected utility with the true distribution $P^0$. First we observe that \textit{ex ante} optimal contingent claims for given beliefs $P_1, \ldots, P_I$ cannot be true Pareto optimal for any $P^0$.

\textbf{Theorem 1.} If the economy contains two individuals $i$ and $j$ such that for some $t$ and some path $\sigma$, $P^i_t(\sigma) \neq P^j_t(\sigma)$, then no \textit{ex ante} true-Pareto optimal allocation can be optimal for any true distribution $P^0$.

\textbf{Proof.} If $P^i_t(\sigma) \neq P^j_t(\sigma)$, then there must exist some other path $\sigma'$ such that $P^i_t(\sigma')/P^j_t(\sigma') \neq P^i_t(\sigma)/P^j_t(\sigma)$, else probabilities cannot sum to one. The first-order conditions for optimality on path $\sigma$ imply that

$$\frac{u^i_t(c^i_t(\sigma))}{u^j_t(c^j_t(\sigma))} = \frac{\lambda_i \beta^i_t P^j_t(\sigma)}{\lambda_j \beta^j_t P^i_t(\sigma)},$$

where the $\lambda$’s, multipliers for the Pareto optimization problem, are both positive. Suppose now that the allocation is true Pareto optimal for some $P^0$. Then first-order conditions imply that there will be positive multipliers $\gamma_i$ and $\gamma_j$ such that

$$\frac{u^i_t(c^i_t(\sigma))}{u^j_t(c^j_t(\sigma))} = \frac{\gamma_i \beta^i_t}{\gamma_j \beta^j_t},$$

Consequently the vectors $(\gamma_i \beta^i_t, \gamma_j \beta^j_t)$ and $(\lambda_i \beta^i_t P^i_t(\sigma), \lambda_j \beta^i_t P^j_t(\sigma))$ are proportional.

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3The assumption of “common knowledge of prior beliefs” is often used, following Aumann [1976], to justify common beliefs. Common prior arguments are critically discussed in Morris (1975). To his analysis we add that the entire apparatus of belief about beliefs about beliefs is simply misplaced in models of trade in large anonymous markets, wherein one individual may have no idea who or what is on the other side of his transaction.
Now consider path $\sigma'$. Since the allocation is truly optimal, it must be the case that:

$$\frac{u'_i(c'_i(\sigma'))}{u_j(c'_j(\sigma'))} = \frac{\gamma_i \beta_j^t}{\gamma_j \beta_i^t} = \frac{\lambda_i \beta_j^t P_t^j(\sigma)}{\lambda_j \beta_i^t P_t^i(\sigma)}.$$ 

Since the allocation is also \textit{ex ante} optimal:

$$\frac{u'_i(c'_i(\sigma'))}{u_j(c'_j(\sigma'))} = \frac{\lambda_i \beta_j^t P_t^j(\sigma')}{\lambda_j \beta_i^t P_t^i(\sigma')}.$$ 

Thus $P_t^i(\sigma')/P_t^j(\sigma') = P_t^i(\sigma)/P_t^j(\sigma)$, which is a contradiction. \hfill \Box

When discount factors are identical, there is in fact a simple necessary condition for true Pareto optimality: Everyone’s consumption is bounded away from 0.

\textbf{Corollary 1.} If individuals have identical discount factors, if the allocation $c$ is true-Pareto optimal, and if for all $i$, $c^i \neq 0$, then for each individual $i$ and all $\sigma$, $\lim \inf_t c^i_t(\sigma) > 0$.

\textit{Proof.} This follows from the fact that the first order conditions are independent of $P^0$, that the welfare weights are positive, and that aggregate endowments are uniformly bounded above and below across paths. \hfill \Box

Another necessary condition for true optimality is that there is no speculation on irrelevant states (frivolous uncertainty).

\textbf{Corollary 2.} Suppose that $c$ is true-Pareto optimal and that the endowment allocation at date $t$ is constant on some event $E$, that is, for $\sigma, \sigma' \in E$, $e_t(\sigma) = e_t(\sigma')$. Then for all individuals $c^i_t(\sigma) = c^i_t(\sigma')$.

\textit{Proof.} Since the allocation is true-Pareto optimal, it must be the case that:

$$\frac{u'_i(c'_i(\sigma))}{u_j(c'_j(\sigma))} = \frac{\gamma_i \beta_j^t}{\gamma_j \beta_i^t} = \frac{u'_i(c'_i(\sigma'))}{u_j(c'_j(\sigma'))}, \quad \forall i, j.$$ 

Then $e_t(\sigma) = e_t(\sigma')$ and the fact that the allocation $c$ is feasible imply the desired result. \hfill \Box
Theorem 1 suggests that the introduction of some kind of market incompleteness could be welfare-improving, that is, incomplete markets could yield allocations that true-Pareto dominate the complete-markets allocation. Unfortunately, the mechanism design problem depends critically on the true distribution $P^0$. Since individuals in the market do not have privileged knowledge of the true distribution, it would be unreasonable to assume that the market designers would have any better knowledge. That is, we want to do distribution-independent parameter design.

Our solution to this problem is to explore the parameter space. We show that there are market institutions which outperform complete markets over much of the parameter space. “Outperform” here has three meanings. For the market interventions we consider we delineate through simulation regions of the model’s parameter space where the intervention is true Pareto superior, where it is better according to a Rawlsian welfare aggregator, and better according to a Bergson-Samuelson social welfare function where the welfare weights are those which solve the *ex ante* Pareto optimality problem.

Finally, we point out that someone whose beliefs are correct cannot be *ex ante* hurt by any Pareto improvement with respect to the true distribution. Consequently, someone with bad beliefs must be made *ex ante* worse off by the change. Since everyone thinks they are right, no one thinks they will be *ex ante* worse off. This is an interesting political economy point.

### 3.3 Spurious unanimity: other approaches

Others have addressed the problem of spurious unanimity in contingent claims allocations. Brunnermeier et al. [2013] introduce *belief-neutral* Pareto optimality. They identify a set of “reasonable beliefs”, potential true distributions, which is the convex hull of the set of individuals’ beliefs. Allocation $x$ is then belief-neutral Pareto superior to allocation $y$ if $x$ is true Pareto superior to $y$ for every true distribution in the set of reasonable beliefs. The intersection of a collection of Pareto orders is, generally speaking, incredibly incomplete. Brunnermeier et al. [2013] reduce incompleteness by examining partial orders induced by Bergson-Samuelson social welfare functions, taking weighted averages of each profile of true expected utilities.

Gilboa et al. [2012] offer a somewhat complicated alternative. Allocation $x$ *no-bet* Pareto improves upon $y$ if $x$ *ex ante* Pareto improves upon $y$ and...
if there exists a potentially true probability distribution such that each individual whose position is *ex ante* improved in the move from $y$ to $x$ also truly prefers $x$ to $y$. This is a direct attempt to remove from Paretian calculations the speculative component to trade that is introduced when beliefs disagree.

The no-bet Pareto relation, while acyclic, can be intransitive. Thus (as they observe), $x$ may no-bet Pareto improve upon $y$, $y$ may no-bet Pareto improve upon $z$, and yet while $x$ *ex ante* Pareto improves upon $z$, this Pareto improvement may be a consequence of spurious contagion.

These two proposals delineate the tradeoffs that arise when considering potential true distributions. Requiring Pareto improvement with respect to a large class of potential true distributions for all welfare comparisons thickens the contract curve; few welfare comparisons can be made. Relaxing this ordinal uniformity condition, however, and allowing different distributions for different comparisons, will, generally speaking, introduce intransitivities.

We do not see any particularly compelling way to undertake welfare analysis when beliefs are heterogeneous. This includes *ex ante* Pareto optimality. So in this paper we carry out the more limited task of identifying sets of potentially true distributions for which given market restrictions are in some sense welfare-improving in some simple examples. We believe that if, in a carefully calibrated model of economic activity, for some market restriction the set of potentially true distributions for which it is a welfare improvement is large, then there is a strong *prima facie* case for introducing it.

## 4 Financial markets, competitive equilibria

In this section we describe optimization problems of an agent under different financial market designs. The first and the key market design is complete financial markets.

*Definition.* The complete financial markets (CM) design is a set of financial markets where market $j$ trades an Arrow security that pays one unit of consumption good next period if state $j$ realizes. Trading is subject to natural borrowing limits defined below.

In the above design, financial markets are maximally unrestricted. On the contrary, in financial autarky, all financial markets are closed.

*Definition.* In the financial autarky (Aut) all financial markets are closed, and all agents consume their endowment.
Financial autarky is the extreme form of financial regulation. While it is impossible to imagine such a design being implemented in practice, it could be superior to the complete markets financial design, as we explain in section 7. In addition to these two extremes, we analyze several other designs: complete markets with borrowing limits (CMB), complete markets in which transactions are taxed (CMT), and markets trading only a risk-free bond subject to a borrowing limit (B). We think of these intermediate designs as partially regulated financial markets and aim to shed light on the relative desirability of different restrictions.

5 A welfare criterion

In what follows we illustrate our arguments using a series of numerical examples. For the purpose of simulations, we define the social welfare function as follows.

Definition. Let $\mathcal{B}$ be a set of admissible beliefs and let $\mathcal{P} = (P^1, ..., P^I) \in \mathcal{B}^I$ denote a belief assignment. Let $P^0 \in \mathcal{B}^0$ be a data generating process, where $\mathcal{B}^0$ is a set of admissible data generating processes. Let $c(\mathcal{P}|\mathcal{M})$ be a competitive equilibrium allocation under a financial market structure $\mathcal{M}$ and a belief assignment $\mathcal{P}$. Then the social welfare function is:

$$\min_{P^0 \in \mathcal{B}^0} \min_{\mathcal{P} \in \mathcal{B}^I} \min_{i} \left[ U_{i,P^0}(c(\mathcal{P}|\mathcal{M})) \right].$$

(4)

This welfare criterion makes three choices, each corresponding to one of the minimization operators. First, the designer chooses a financial market structure that benefits the least-advantaged members of society. This is one of the two principles of justice proposed in Rawls [1971]. Alternatively we could specify social welfare as a weighted sum of utilities of different groups. However, any set of weights would be arbitrary. Our paternalistic designer is spared the obligation of deciding a fair set of weights.

Second, many configurations of beliefs are possible. For example, some agents might be optimistic and undertake excessively risky investments that could drive them quickly out of the financial markets. Financial restrictions seem desirable in this case. On the other hand, pessimistic agents might over-invest in safe assets. They would still be driven out of a complete financial market, but perhaps at a slower rate. Financial regulations in this case would have to strike a balance between rescuing agents from financial ruin.
and limiting insurance opportunities. The analysis would be more involved if optimistic and pessimistic agents were both present. The possibilities are limitless, and each may support a different financial market design. For this reason, we consider only the worst possible assignment of beliefs.

Third, many realistic stochastic processes for consumption – processes with e.g. long-run risks as in Bansal and Yaron [2004] and disaster risk as in Rietz [1988] and Barro [2006] – are difficult to distinguish from a random walk. So, we do not grant our designer knowledge of the true “belief” $P^0$. Instead the designer chooses a financial market structure that would provide a satisfactory welfare level even under the worst possible assignment of the data generating process.

Our choices are, in part, motivated by the analysis in Rawls [1971] who argues that a fair social choice can only be made in a hypothetical “original position”:

No one knows his place in society, his class position or social status, nor does anyone know his fortune in the distribution of natural assets and abilities, his intelligence, strength, and the like. I shall even assume that the parties do not know their conceptions of the good or their special psychological propensities. The principles of justice are chosen behind a veil of ignorance.

For our purposes, we replace “principles of justice” with “design of financial markets.” The veil of ignorance advocated by Rawls allows devising a set of rules that are independent of the current economic fundamentals – beliefs assignment, true data generating process, and wealth distribution.

The main advantage of this criterion is that it does not require that the designer know the true probabilities $P^0$ and configuration of beliefs $P$. This criterion only requires the designer to consider a sufficiently large set of beliefs $B$ and data generating processes (dgp) $B^0$. Notice also that the criterion allows for $B \cap B^0 \neq \emptyset$.

Finally, it is important to consider a restricted set of belief assignments. Had we removed assumption A4 our criterion would select beliefs that assign all probability to the worst path.

5.1 Complete markets economy

We start with a formulation of the optimization problem under dynamically complete markets. Let $Q_t(\sigma)$ be the date-$t$ price of an Arrow security that
pays along path \( \sigma \). The number of Arrow securities purchased by a type-\( i \) agent in period \( t \) along history \( \sigma \) is denoted by \( a^i_t(\sigma) \). Then a type-\( i \) agent faces the following budget constraint:

\[
\tilde{c}^i_t(\sigma) + \sum_{\tilde{\sigma} | \sigma^t} Q_{t}(\tilde{\sigma}) a^i_{t+1}(\tilde{\sigma}) = a^i_t(\sigma) + \tilde{c}^i_t(\sigma).
\] (5a)

Purchases of Arrow securities are subject to natural borrowing limits:

\[
a^i_{t+1}(\sigma) \geq -N^i_{t+1}(\sigma),
\] (5b)

which are constructed as follows. Define the \( j \)-period ahead price \( Q^i_j(\sigma) = \Pi_{k=0}^{j-1} Q_{t+k}(\sigma) \). Then a natural borrowing limit equals the date-\( t \) value of the continuation of an agent’s endowment plan:

\[
N^i_t(\sigma) = \sum_{j=0}^{\infty} \sum_{\tilde{\sigma} | \sigma^t} Q^i_j(\tilde{\sigma}) \tilde{c}^i_{t+j}(\tilde{\sigma}).
\] (6)

Natural borrowing limits never bind in a competitive equilibrium if a period utility function satisfies an Inada condition \( A2 \). A type-\( i \) agent chooses consumption and asset trading plans to maximize life-time utility (1) subject to constraints (5a) and (5b).

Finally, we define the prices of two assets to which we refer later. Price of a risk-free bond is:

\[
q^b_t(\sigma) = \sum_{\tilde{\sigma} | \sigma^t} Q_{t}(\tilde{\sigma}).
\] (7a)

The price of a claim to the aggregate endowment is:

\[
q^e_t(\sigma) = \sum_{j=0}^{\infty} \sum_{\tilde{\sigma} | \sigma^t} Q^j_{t}(\tilde{\sigma}) e_{t+j}(\tilde{\sigma}).
\] (7b)

### 5.2 Bond economy

**Definition.** A bond-only financial market design (B) consists of a single market that trades a risk-free bond subject to an exogenous borrowing limit.

In the bond economy, a type-\( i \) agent faces the following constraints:

\[
c^i_t(\sigma) + q^b_t(\sigma) a^i_{t+1}(\sigma) = a^i_t(\sigma) + c^i_t(\sigma),
\] (8a)

\[
a^i_{t+1}(\sigma) \geq -B^i_{t+1}(\sigma),
\] (8b)
where \( q_t^i(\sigma) \) denotes the date-\( t \) price of a risk free bond, \( b_t^i(\sigma) \) represents the date-\( t \) bond purchases of agent \( i \), and \( B_{t+1}^i(\sigma) \) is an \textit{exogenous} borrowing limit. These borrowing limits have to be sufficiently tight to make sure that all loans are repaid with certainty. Borrowing limits must be tighter than the worst-case date-\( t \) value of the continuation of an agent-\( i \)'s endowment plan:

\[
\inf_{\tilde{\sigma} | \sigma} \left[ e_t^i(\tilde{\sigma}) + \sum_{j=0}^{\infty} \Pi_{j=0}^{j-1} q_t^i k_{t+k}^i (\tilde{\sigma}) e_t^i (\tilde{\sigma}) \right].
\]

The above borrowing limit is the largest limit that can (potentially) be imposed on a type-\( i \) after history \( \sigma_t \) agent in the bond-only economy. However, unlike in the complete markets economy, an endogenous borrowing limit cannot be determined before solving for a competitive equilibrium. Hence, an exogenous borrowing limit must be imposed instead.

### 6 Illustrative examples

We now present our leading example, which we use to demonstrate economic forces that operate in the economies with heterogenous beliefs. We show how to apply our welfare criterion and use it to compare the complete markets and the risk-free bond financial market designs.

Consider the following economy. There are two types of agents and three states: \( \sigma_t \in \{0, 1, 2\} \). The economy begins in state 0 and then exits to states 1 and 2.\(^4\) Endowments are specified as follows:

\[
(e_t^1(\sigma), e_t^2(\sigma)) = \begin{cases} (0.5, 0.5) & \text{if } \sigma_t = 0 \\ (e_h, e_l) & \text{if } \sigma_t = 1 \\ (e_l, e_h) & \text{if } \sigma_t = 2 \end{cases}, \forall t, \sigma. \quad (9)
\]

So, there is no aggregate uncertainty but individuals face idiosyncratic risk. Beliefs are specified as follows:

\[
\Pi^i = \begin{bmatrix} 0 & 0.5 & 0.5 \\ 0 & p^i & 1 - p^i \\ 0 & p^i & 1 - p^i \end{bmatrix}, \quad (10)
\]

\(^4\)The only purpose of the transitory state 0 is symmetry. It insures that agents begin with identical endowments and can trade prior to the realization of states 1 and 2.
where $\Pi^0$ denotes the true probability transition matrix. Subjective probabilities over histories $P_t^i(\sigma)$ are computed using individual transition matrices.

First, we describe a competitive equilibrium in the complete markets economy when beliefs are homogeneous. Beliefs do not have to be correct. Because there is no aggregate uncertainty and preferences are homothetic, both agents consume a constant amount. The competitive equilibrium allocation is:

$$
(c_1^t(\sigma), c_2^t(\sigma)) = (0.5 + \beta^2(\mu_e - 0.5), 0.5 - \beta^2(\mu_e - 0.5)), \forall t, \sigma, \quad (11)
$$

where $\mu_e \equiv pe_l + (1 - p)e_h$ is the expected endowment evaluated using the common beliefs, $p$. An agent achieves a constant consumption plan by buying $A_j \equiv 0.5 - e_j + \beta(\mu_e - 0.5)$ Arrow securities paying in the state where his income is $e_j$. The quantity of Arrow securities traded in equilibrium, $|A_j|$, is small relative to the natural borrowing limit: $N_i^t(\sigma) = c_i^t(\sigma) + \beta\mu_e/(1 - \beta)$.

Second, we describe a competitive equilibrium in the complete markets with heterogeneous beliefs. Suppose that $p_1 = p^0$ and $p_2 \neq p^0$. As shown in Blume and Easley [2006] consumption of a type-2 agent must converge to zero:

$$
\lim_{t \to \infty} c_2^t(\sigma) = 0, \quad P^0 - a.s. \quad (12)
$$

Following Blume and Easley [2006] we say that type-2 agents do not survive. The survival force is the major force in our analysis. Type-2 agents are being driven out because they make poor financial decisions.

Agents invest in Arrow securities for two reasons: income hedging and disagreement. Suppose $p_2 > p^0$. To hedge income fluctuations, a type-2 agent buys Arrow securities that pay in state 1 (his low income state) and sells Arrow securities that pay in state 2 (his high income state). Yet, because a type-2 agent overestimates probability of state 1, he buys extra securities that pay in this state. So, he over-invests in securities that pay in state 1 and under-invests in securities that pay in state 2. These additional trades are “speculative.” As a result of these trades, a type-2 agent’s consumption increases every time state 1 realizes. The opposite happens if state 2 realizes. Yet, state 1 is less likely than a type-2 agent anticipates. So, as shown by Blume and Easley [2006], his investments pay off less than expected, he loses wealth on average, and his consumption converges to zero.

---

5 Speculation is trading activity that is motivated by differences in beliefs and would be absent had all agents had the same beliefs.
Figure 1 plots 200 sample paths of consumption (panel A) and financial wealth (panel B) of a type-2 agent in the complete markets economy. The solid line in each panel denotes the average across sample paths. Both consumption and wealth drift towards their respective lower bounds. Yet, the speed of convergence is very slow: for example, after 100 years a type-2 agent’s consumption decreases from 0.493 to 0.432 along the average path. The decline in financial wealth is more substantial, falling from 0 to -1.524 (or roughly three average individual annual incomes) along the average path.

Despite a decline in his consumption and financial wealth, a type-2 agent believes that what happens to him is simply bad luck. Figure 2 demonstrates the difference between actual and perceived outcomes. This figure plots expected, from a point of view of type-2 agent, evolution of his consumption and financial wealth in periods 51-100 assuming that during periods 0-50 he followed the “average path.” Not surprisingly, he expects to prosper. This is a manifestation of another result in Blume and Easley [2006]:

$$\limsup_{t \to \infty} c_t^i(\sigma) = 1, \quad P\text{-}a.s.$$

Finally, we present welfare levels for the two types of agent. As a benchmark, we begin by calculating welfare in the complete markets economy when
beliefs are homogenous and coincide with the truth. Assuming $p^0 = 0.50$, this benchmark level of welfare, denoted by $W^*$, is $-2$ for each type. Subjective welfare levels in the heterogeneous beliefs economy are $-1.943$ and $-2.124$ respectively for a type-1 and a type-2 agent. A type-1 agent, whose beliefs coincide with the truth, expects higher welfare than $W^*$. He is better off in the economy with diverse beliefs as his “speculative” financial strategy allows him to accumulate wealth. A type-2 agent expects welfare level that is lower than $W^*$. This happens because endowment plan of a type-2 agent is valued less. The reality is similar. Objective welfare levels are $-1.947$ and $-2.129$ respectively for a type-1 and type-2 agent.\(^6\) Belief diversity has substantial impact on welfare: relative to the common beliefs benchmark a reduction in a type-2 agent’s welfare is equivalent to a permanent 6.45% decline in his consumption.\(^7\) So, welfare of a type-2 agent, and hence the society’s welfare, in this economy is low. Two sources contribute to this effect: consumption volatility and a downward trend in a type-2’s consumption. To quantify contribution of each source we compute welfare of a type-2 agent

\(^6\)Note that luckily a type-2 agent’s objective and subjective welfare levels are very similar.

\(^7\)Cost of aggregate fluctuations in a standard RBC model is typically found to be below 0.1%.
along the “average path” and it is \(-2.091\). Thus, low welfare of a type-2 agent is caused largely by a diminishing trend in his consumption rather than by increased consumption volatility.\(^8\)

## 6.1 Bond economy

We now describe the bond-only design. In the bond-only economy agents can save and borrow by buying or selling bonds, but they cannot transfer income across states. We can think of the bond-only design as the one in which there are restrictions on the set of payoffs that assets can span. While it is the more stringent restriction we also must impose a borrowing limit as explained in section 5.2. Since it is impossible to devise \textit{a priori} a borrowing limit that would never bind we impose an exogenous, yet generous, limit of 16 average individual annual incomes: \(B_i(\sigma) = 8, \forall t, \sigma\).

Returning to numerical example 1, we simulate equilibrium consumption and wealth dynamics in the bond economy. As shown in figure 3, consumption and financial wealth for type-2 agents now grow on average. Consumption increases from an average of 0.492 to 0.526 (panel A), and financial wealth rises from an average of 0 to 0.878, or 1.76 average individual annual incomes (panel B). As explained in Cogley et al. [forthcoming], this happens because a type-2 agent is pessimistic and buys bonds as a precautionary store of value. The real interest rate falls relative to that in a homogenous-beliefs economy, and the lower real rate makes type-1 agents willing lenders.

Subjective welfare levels are \(-2.004\) and \(-2.011\), respectively, for a type-1 and a type-2 agent. Both agents expect to be worse off than in the benchmark economy in which all agents have the same correct beliefs. Objective welfare levels show that a type-2 agent, despite accumulating financial wealth, has lower welfare. Pessimism drives a type-2 agent to postpone consumption far into the future. Even after accounting for the lower real interest rate, this delayed gratification lowers expected utility.

\(^8\)It is natural to ask what would happen in this economy if a type-2 agent were optimistic. To answer this we studied the case with \(p^0 = p^1 = 0.50, p^2 = 0.45\). Welfare levels in this case are: \(U_{p0}^1 = U_{p1}^1 = -2.002, U_{p0}^2 = -2.063\) and \(U_{p2}^2 = -2.058\). That is a type-2 agent still has the lowest welfare in the economy but it is not as low. This happens largely because optimism increases the value of his endowment plan. So, his consumption while decreasing on average starts from a value above 0.5. If we replaced his consumption plan with an average plan his welfare would be \(-2.024\). So, here the welfare loss is attributed mainly to increased consumption volatility. See also section A.1.
Figure 3: Sample paths of consumption and financial wealth of a type-2 agent in the bond economy. Parameters: $\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = p^1 = 0.5, p^2 = 0.55, B = 8$.

### 6.2 Bond-only vs complete markets

If $(p^1 = p^0 = 0.5, p^2 = 0.55)$ were the only admissible beliefs, our welfare criterion would select the bond-only design over the complete markets design. The former delivers a substantial welfare level to both types because it limits speculation, but still allows resources to be transferred across periods. Under the latter, type-1 agents take advantage of the poor forecasting abilities of type-2 consumers, eventually driving them to destitution.

Matters are more complicated when we consider a larger set of admissible beliefs. For instance, suppose $(p^1, p^2) \in [0.45, 0.55]^2$, holding $p^0$ fixed at 0.5. Figure 4 plots the welfare surface $\min_i [U_{p0}(c(p^1, p^2|M))]$ for this belief set. The lowest welfare level under the bond-only design is $-2.011$, and it is achieved at $(p^1, p^2) = (0.45, 0.45)$ and $(0.55, 0.55)$. At these “critical points” (depicted by black points in the figure), beliefs are homogeneous but wrong. Under these belief assignments, one of the two types is commonly believed to have an inferior endowment plan.

The lowest welfare in the complete markets economy is $-2.139$, and it is achieved at $(p^1, p^2) = (0.45, 0.525)$ and $(0.475, 0.55)$ (portrayed by gray points in the figure). At the critical points, beliefs are close to being maximally different. Consider the belief assignment $(p^1, p^2) = (0.45, 0.525)$. At this
Figure 4: Welfare in example 1: the bond-only (black) vs the complete markets (grey) design. Square point denotes the unconstrained maximum: $(p^1, p^2, W^*) = (0.5, 0.5, -2)$. Circle points denote belief assignments that attain the lowest welfare under the corresponding design.

Parameters: $\beta = 0.96, \epsilon_l = 1/3, \epsilon_h = 2/3, p^0 = 0.5, B = 8.$

point a type-1 agent has lower welfare. Two forces are acting against him. First, his beliefs are less accurate, and, so, his consumption is eventually driven to zero. Second, he is more pessimistic than a type-2 agent, and his endowment stream is valued less – he is subject to a negative wealth effect. But type-2 consumers are also pessimistic, and this activates a wealth effect that reduces a type-2 agent’s and, therefore, society’s welfare. Thus, at the considered belief assignment, survival forces are not as strong as they could be.

In this example the bond-only design is preferred over the complete markets design because:

$$-2.011 = \min_{p^1, p^2} \min_i U^i_{p^0}(c^i(\mathcal{P}|B)) > \min_{p^1, p^2} \min_i U^i_{p^0}(c^i(\mathcal{P}|CM)) = -2.139.$$ 

The complete markets design would be preferred if the set of admissible beliefs were reduced to $[0.49, 0.51]^2$. That is, belief heterogeneity must be
restricted substantially to support complete markets. It is surprising, though, that the bond-only design performs so robustly when there is no aggregate risk.

6.3 Effects of time preference and disagreement

The choice of financial design also depends on the discount factor \( \beta \). To illustrate the effect of time preference, we fix \( p^1 = p^0 = 0.5 \) and specify the admissible belief set as \( p^2 \in [0.45, 0.55] \). Then we let the common discount factor \( \beta \) vary between 0.8 and 0.99. Figure 5 plots the social welfare surface under the bond-only (black) and complete markets (gray) designs.\(^9\)

\[\text{Figure 5: Welfare in example 1: the bond-only (black) vs the complete markets (gray) design. Circle points denote belief assignments that attain the lowest welfare under the corresponding design when } \beta = 0.99.\]

Parameters: \( e_l = 1/3, e_h = 2/3, p^0 = p^1 = 0.5, B = 2.\)

\(^9\)Note that the borrowing limit under the bond-only design was tightened the borrowing limit under the bond-only design so that we could study preferences with a discount factor as low as 0.8.
As the discount factor increases, the minimum welfare in the bond economy dominates the complete markets economy on a larger set of belief specifications. This happens because agents care more about the limiting behavior of their consumption plans when they are more patient, making the complete markets design unattractive even if disagreement is small. For instance, for $\beta = 0.99$, society’s welfare is -2.637 and -2.008, respectively, under the complete markets and the bond-only designs. In this case, restricting financial markets to trade only a risk-free bond is equivalent to a permanent 31.3% increase in consumption! The more patient the agents are, the less appealing is the complete markets design.

The degree to which beliefs can differ also affects market design. Suppose the set of admissible beliefs were expanded. Under the complete markets design, consumption of the least advantaged type-2 agent would decrease faster and his welfare fall by more as the maximum degree of disagreement increased. Hence the complete markets structure becomes less attractive as the belief set expands. In contrast, the bond economy still tames the survival forces, and consumption of type-2 agents does not differ much from 0.5. This makes the bond-only design a more attractive choice according to our criterion.

Propositions 2 and 3 in section 7 formalize our intuition on the effects of disagreement and time preference.

### 6.4 Borrowing limits

In this section we describe the design in which a rich set of assets is traded but trading is subject to an exogenous borrowing limit $B$ that is tighter than the natural borrowing limits in (5b):

$$a_{i+1}\sigma \geq -B. \quad (13)$$

When borrowing limits tighter than the natural borrowing limit are enforced, all agents survive even when markets are complete. Thus, the complete markets with borrowing limits is an alternative design that tames the survival forces of Blume and Easley [2006].

We continue to assume that $p^0 = 0.50$ is the true probability of state 1 and that the admissible set of belief assignments is $(p^1, p^2) \in [0.45, 0.55]^2$. We set $B = 1$, which is equivalent to two average individual annual incomes. Figure 6 shows the social welfare surface for this environment (black) and contrasts it with the benchmark complete markets design (gray).
Figure 6: Welfare in example 2: the complete markets with borrowing limits (black) vs the complete markets design (gray). Square point denotes the unconstrained maximum: \((p_1^*, p_2^*, W^*) = (0.5, 0.5, -2)\). Circles denote belief assignments that attain the lowest welfare under the corresponding design. Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 1\).

The square depicts the maximum achievable welfare in the two economies. It is reached at \((p_1^*, p_2^*) = (0.5, 0.5)\) in both cases and is equal to \(W^* = -2\). When agents agree, there is little trading, and borrowing limits are slack. The benchmark complete markets design is superior when disagreement is small, and the design with borrowing limits is preferred at higher levels of disagreement.

The two circles portray the minimum welfare achieved under the respective market designs. Under the design with borrowing limits, the lowest welfare levels are achieved at either \((p_1^*, p_2^*) = (0.45, 0.48)\) or \((p_1^*, p_2^*) = (0.53, 0.55)\). As in the bond economy, belief heterogeneity ceases to be the critical force defining the lowest welfare in the economy. Instead, at the critical belief assignments, agents nearly agree on one of the types being poor. For example, at the point \((p_1^*, p_2^*) = (0.45, 0.48)\) everyone agrees that type-1 agent is less likely to receive high endowments. Moreover, a type-1 agent’s
beliefs are less accurate. For both reasons, his and the society’s welfare is lower. At \((p^1, p^2) = (0.53, 0.55)\) it is a type-2 agent who suffers. Tightening the borrowing limit significantly lessens speculation and, therefore, survival forces. For this example, society’s welfare increases from -2.139 to -2.083, a difference equivalent to a 2.7% permanent increase in consumption.

Next we turn to the economy in which the type-1 agent knows the truth and the type-2 agent is pessimistic, \((p^1, p^2) = (0.50, 0.55)\). We compute mean and standard deviation (in parentheses) of the following variables: type-2 agent’s financial wealth \(a_t^2(\sigma)\), his consumption \(c_t^2(\sigma)\) and prices of a risk-free bond \(q_t^b(\sigma)\) and a claim to the aggregate endowment \(q_t^e(\sigma)\) (see (7) for definition).\(^{10}\) We contrast two designs: complete markets with (restrictive) \(B = 1\) and (relaxed) \(B = 8\) borrowing limits. Table 1 summarizes our findings. First, financial wealth of type-2 agent stays closer to 0 under the tighter borrowing limit \(B = 1\). Financial wealth is also 3.79 times less volatile than under \(B = 8\)! Borrowing limits do not allow a type-2 agent to be impoverished. They stop short of financial ruin awaiting for him under the complete markets design. Moreover, he is often lucky to accumulate the maximal financial position \(a_1^1 = 1\). Second, consumption of a type-2 agent stays closer to 0.5 and it is also 2.43 times less volatile! A more even and less volatile distribution of consumption is the source of welfare gains in the design with the tight borrowing limit. Third, observe that prices of the two financial assets are increased and they are also more volatile. That is by tightening the borrowing limit the designer drives volatility out of consumption and into prices. This suggests that financial price stability may conflict with welfare maximizing policies.

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<th>(\ln(q^b))</th>
<th>(\ln(q^e))</th>
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<td>0.517</td>
<td>-0.037</td>
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<td>(0.612)</td>
<td>(0.038)</td>
<td>(0.008)</td>
<td>(0.030)</td>
</tr>
<tr>
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<td>(2.317)</td>
<td>(0.092)</td>
<td>(0.002)</td>
<td>(0.022)</td>
</tr>
</tbody>
</table>

Table 1: Mean and standard deviation (in parentheses) for the complete markets with borrowing limit design

\(^{10}\) We simulated 11,000 periods starting from a random state and \((a_0^1, a_0^2) = (0, 0)\). We discarded the first 1,000 observations.
6.5 Transaction tax

Last but not least, we describe a design in which a full set of Arrow securities is traded but trading is subject to a transaction tax. As before, trade is subject to an exogenous borrowing limit $B = 8$ (sixteen average individual annual incomes), but we replace budget constraint (5a) with the following:

$$c_i^t(\sigma) + \sum_{\tilde{\sigma}|\sigma^t} Q_t(\tilde{\sigma})a_i^{t+1}(\tilde{\sigma}) + \tau \cdot \sum_{\tilde{\sigma}|\sigma^t} [a_i^{t+1}(\tilde{\sigma}) - a_i^t(\sigma)]^2$$

$$= a_i^t(\sigma) + e_i^t(\sigma) + T_t(\sigma)/2,$$

(14)

where $T_t(\sigma)$ is the total transaction tax revenue. We made two assumptions when setting up this financial design. First, we chose the transaction tax to be a quadratic function of security purchases. This is rather a technical assumption that insures continuity of demands for securities. Second, we chose to rebate the transaction tax back to investors.

As before, trade is subject to an exogenous borrowing limit $B = 8$ (sixteen average individual annual incomes).

The transaction tax reduces demand of each agent and pushes the economy towards no-trade equilibrium. The transaction tax limits trading activity but not necessarily insurance possibilities. Agents are able to transfer income across states but it is costly to do so.

Figure 7 shows welfare under three market designs: complete markets with a natural borrowing limit, complete markets with an exogenous borrowing limit $B = 8$, and complete markets with $B = 8$ plus a transaction tax $\tau = 0.05$. Welfare for the first two designs are very close, suggesting that competitive equilibrium allocations under $B = 8$ are close to allocations under the natural borrowing limit. Imposing a transaction tax on top of the borrowing limit increases society’s welfare from -2.134 to -2.079, an amount equivalent to a permanent 2.6% increase in consumption.

The minimal welfare in society is achieved at either $(p^1, p^2) = (0.55, 0.51)$ or $(p^1, p^2) = (0.49, 0.45)$. For the first belief assignment, a type-2 agent has lower welfare and, therefore, determines the society’s welfare. Note that this happens despite having more accurate beliefs. Both types agree that type-2 agents have a higher probability of receiving low income, however, and the wealth effect drags down their welfare.
Figure 7: Welfare in example 3: complete markets with borrowing limits and transaction tax (black) vs complete markets with borrowing limit (dark gray) vs complete markets. Square point denotes the unconstrained maximum: \((p^1, p^2, W) = (0.5, 0.5, -2)\). Circle points denote belief assignments that attain the lowest welfare under the corresponding market design.

Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 8, \tau = 0.05\).

7 Theoretical results

In this section we present theoretical results that highlight conditions under which restricting financial markets improves social welfare. We start with a very simple observation. Consider the case with “frivolous” uncertainty, which we model as states of the world that do not affect payoffs. For example, the states “high income, sun” and “high-income, rain” are frivolous. Agents could disagree about likelihood of these two states, but payoffs are equal. Our first proposition states that markets trading securities with payoffs contingent on frivolous states should be shut down.

Proposition 1 (Frivolous uncertainty).
Assume that agents have a common discount factor \(\beta\), a common utility
function $u$ and their endowment streams are ex-ante identical. Suppose that agents marginal beliefs over endowments agree. Then restricting agents to purchase the same consumption across all frivolous states cannot reduce social welfare and may improve it.

Proof. When the trade on frivolous states is banned the competitive allocation is $c^i = e/I, \forall i$. This allocation is Pareto optimal with respect to the true dgp and therefore achieves the unconstrained optimal level of society’s welfare. All other plans are obviously inferior. When markets for frivolous uncertainty are open the competitive allocation may deviate from the unconstrained optimum. In particular consumption of a type-$i$ and a type-$j$ agent depends on the relative likelihood they assign to the path $\sigma$. Therefore, if agents disagree, then social welfare must decrease.

Our second result, proposition 2, shows that if the set of belief assignments is sufficiently large then even financial autarky will be preferred to the complete markets design. The proof that follows highlights how disagreement can lead to very low consumption on some paths.

Proposition 2 (Autarky may dominate the complete markets design).
Assume that agents have a common discount factor $\beta$, a common utility function $u$ that is unbounded below, their endowment streams are ex-ante identical and $\inf e^i(\sigma) > \varepsilon > 0$. Then there exists a set of belief assignments $B$ such that:

$$\min_{P^0_0} \min_{(P^0_1,\ldots,P^0_I) \in B^I} \min_i U^i_{P^0_0}(c^i(P|CM)) < \min_{P^0_0} \min_{(P^0_1,\ldots,P^0_I) \in B^I} \min_i U^i_{P^0_0}(c^i(P|Aut)).$$

Proof. Without loss of generality assume that there are only two types of agents. Under the complete markets design the competitive equilibrium allocation satisfies the following system of equilibrium conditions:

$$\frac{P^1_t(\sigma)u^i(c^1_t(\sigma))}{P^2_t(\sigma)u^i(c^2_t(\sigma))} = \theta, \quad \forall t, \sigma,$$

where $\theta$ is a positive constant. Let $t = 1$ and fix $\sigma$ with $P^0_1(\sigma) > 0$. Then the society’s welfare is bounded above by the following quantity:\textsuperscript{11}

$$U^i_{P^0_0} \leq (1 - \beta)[u(F) + \beta(P^0_1(\sigma)u(x) + (1 - P^0_1(\sigma))u(F)) + \beta^2 u(F)] \equiv \bar{U}(x),$$

\textsuperscript{11}That is we give an agent $i$ all endowment in all periods and on all paths except for period 1 on path $\sigma$. 

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where $x$ solves the following equation:

$$\theta^{-1} \frac{P_1^t(\sigma)}{P_2^t(\sigma)} = \frac{u'(\bar{e}_1(\sigma) - x)}{u'(x)}.$$ 

Further analysis is complicated by the fact that $\theta$ in general depends on assignment of beliefs. (This is a manifestation of the wealth effect discussed above.) To deal with this issue we exploit our assumption that endowments are ex-ante identical. That is for any $t, \sigma$ there exists $\sigma'$ such that $(e^1_t(\sigma), e^2_t(\sigma)) = (e^2_t(\sigma'), e^1_t(\sigma'))$. Then suppose that $P^1_t(\sigma'') = P^2_t(\sigma''), \forall t \neq 1, \sigma'' \notin \{\sigma, \sigma'\}$. That is agents disagree only in period 1 and only on the two paths where their endowments are symmetric opposites. When agents disagree we insist that their beliefs are: $P^1_1(\sigma) = P^2_1(\sigma'), P^1_1(\sigma') = P^2_1(\sigma)$. (Appendix A.2 provides an example with “symmetric” beliefs.) With such a choice of beliefs $\theta = 1$ as agents are ex-ante identical, both with respect to their endowments and their beliefs.

We now show that consumption of one of the agents can be arbitrarily small if we increase disagreement. As $P^1_t(\sigma)/P^2_t(\sigma)$ converges to zero by assumption A2 and $u'(x)$ converge monotonically to 0 and $+\infty$ respectively. Because $u(\bar{e}) > -\infty$ we can decrease $P^1_t(\sigma)/P^2_t(\sigma)$ until:

$$\bar{U}(x) < u(\bar{e}) \leq \min_{p^i \in \mathcal{P}} \min_i U^i_{p^i}(e^i).$$

Thus, as long as the set of admissible beliefs $\mathcal{B}$ contains the belief assignment constructed above, our welfare criterion selects the financial autarky design over the complete markets. 

Our third result demonstrates that if we make agents sufficiently patient then the financial autarky will again dominate the complete markets design. This result, however comes at a cost – we only consider symmetric assignments of beliefs, i.e. for every $t$ and $\sigma$ there is a $\sigma'$ such that $(P^1_t(\sigma), P^2_t(\sigma)) = (P^2_t(\sigma'), P^1_t(\sigma'))$. This is needed to disable the wealth effect.

**Proposition 3** (Patient agents prefer restricted financial designs).

Assume that agents have a common discount factor $\beta$, a common utility function $u$, their endowment streams are ex-ante identical and $\inf e^i_t(\sigma) >$
\( \epsilon > 0 \). Further, assume that the set of admissible beliefs assignments includes only symmetric beliefs. Then there exists \( \bar{\beta} < 1 \) such that for all \( \beta \in [\bar{\beta}, 1) \):

\[
\min_{P^0} \min_{(P^1, \ldots, P^I) \in B^I} \min_i \left( \min_{\mathcal{P} \in \mathcal{P}(CM)} U^i_{P^0}(c^i(\mathcal{P} | CM)) \right) < \min_{P^0} \min_{(P^1, \ldots, P^I) \in B^I} \min_i \left( \min_{\mathcal{P} \in \mathcal{P}(Aut)} U^i_{P^0}(c^i(\mathcal{P} | Aut)) \right).
\]

**Proof.** Without loss of generality assume that there are only two types of agents. Because beliefs are symmetric, under the complete markets design the competitive equilibrium allocation satisfies the following system of equilibrium conditions:

\[
\frac{P^1_t(\sigma)u'(c^1_t(\sigma))}{P^2_t(\sigma)u'(c^2_t(\sigma))} = 1, \quad \forall t, \sigma.
\]

Importantly, the competitive allocation does not depend on \( \beta \). Select \( P^0 \) such that agent 1 has more accurate beliefs. Then the analysis in Blume and Easley [2006] shows that for any \( \epsilon > 0 \) there exists \( T \) such that \( c^2_t(\sigma) \leq \epsilon, \forall t \geq T \) almost surely given \( P^0 \). This gives us an upper bound on the society welfare:

\[
U^2_{P^0}(c^2) \leq (1 - \beta^T) u(F) + \beta^T u(\epsilon) \equiv U(\beta).
\]

Fix \( \epsilon = \epsilon/2 \). By the intermediate value theorem there exists \( \bar{\beta} \in (0, 1) \) such that \( U(\bar{\beta}) < u(\epsilon), \forall \beta > \bar{\beta} \). Therefore, for all \( \beta > \bar{\beta} \) we have:

\[
\min_{P^0} \min_{(P^1, \ldots, P^I) \in B^I} \min_i \left( \min_{\mathcal{P} \in \mathcal{P}(CM)} U^i_{P^0}(c^i(\mathcal{P} | CM)) \right) < u(\epsilon)
\]

\[
\leq \min_{P^0} \min_{(P^1, \ldots, P^I) \in B^I} \min_i \left( \min_{\mathcal{P} \in \mathcal{P}(Aut)} U^i_{P^0}(c^i(\mathcal{P} | Aut)) \right).
\]

That is the financial autarky dominates the complete markets design if agents are sufficiently patient.

We would like to remark that by imposing endogenous solvency constraints as in Alvarez and Jermann [2000] we can insure that the welfare under the complete markets design is at least as high as in autarky. Therefore, Propositions 2 and 3 could be about the complete markets versus the complete markets with endogenous solvency constraints. But this is the only design with partially restricted markets that we can address theoretically.

Finally, our theoretical predictions would be unchanged if instead of selecting the worst-case assignment of beliefs and the truth we had a uniform prior over those. To understand this note that because we assume that \( u \) is unlimited below there always exist assignments with arbitrarily low welfare under the complete markets design.
8 Market weights

In this section we demonstrate our criterion with market weights. That is for any belief assignment $\mathcal{P} = (P^1, P^2)$ we solve for the competitive equilibrium and compute the implied Pareto weights for type-1 and type-2 agents. Let $\theta^i(\mathcal{P})$ denote the implied (normalized) Pareto weight of type-$i$ agent. Then the social welfare function is defined as follows:

$$W(\mathcal{M}) = \min_{P^0} \min_{\mathcal{P}} \sum_{i=1}^{I} \theta^i(P^0)U^i(P|\mathcal{M}).$$ \hspace{1cm} (15)

This criterion replaces the lowest welfare in the society with a particular weighted average of individual welfare levels. It uses the market weights that are the implied Pareto weights for the complete markets economy with the same assignment of beliefs.\footnote{With logarithmic preferences Pareto weights are wealth shares. That is the weight of agent $i$ is the proportion of the aggregate wealth owned by him. This suggests another possibility that is to use wealth shares from the complete markets competitive equilibrium.} Thus defined welfare criterion has a potential benefit over the Rawlsian one. Under the new criterion choice of allocation cannot be driven by a small but disadvantaged group because its Pareto weight would, in general, be small.

Figure 8 plots the welfare weight of agent 2 for $(p^1, p^2) \in [0.45, 0.55] \equiv \mathcal{B}$. The weight equals 0.5 on the diagonal where agents are symmetric opposites: $p^i = 1 - p^2$. Consider now moving away from the diagonal towards $(p^1, p^2) = (0.45, 0.45)$. Agent 2 becomes more optimistic and agent 1 more pessimistic; so, type-2 agent’s weight increases and type 1 agent’s weight decreases. That is because prices reflect the common belief that type 2 is more likely to receive high endowment. So, type 2 is wealthier and, so, his market weight is higher.\footnote{This is a manifestation of the wealth effect that may sometimes dominate survival/speculative forces.}

We now turn to welfare comparison of the bond-only economy and the complete markets economy. We continue to fix $p^0 = 0.5$. Figure 9 plots the society’s welfare $\sum_{i} [\theta^i(\mathcal{P})U^i(P^0(c^i(\mathcal{P})))) for the two financial markets designs. When beliefs coincide with the truth, $p^1 = p^2 = p^0$, then welfare of each type is -2 – the maximum achievable under any market design (depicted by the gray square point). The society’s welfare is close to this benchmark when agents agree even if they are wrong, i.e. on the diagonal.}
Figure 8: Implied Pareto weights for the complete markets economy.
Parameters: $\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5$.

with $p^1 = p^2$. Close to the diagonal welfare under the bond-only design and under the complete markets design are close. But the latter is higher because disagreement is small and survival forces are weak. So, while agents are being driven out of the financial markets this occurs extremely slowly. When we move away from the diagonal with homogeneous beliefs society’s welfare stays robustly high under the bond-only design but declines quickly under the complete markets design. The reason for such robust performance of the bond-only design is that it keeps survival forces locked. That is differences in beliefs have little effect on the equilibrium outcome when only a risk-free bond is traded. The lowest welfare under the bond-only design is achieved at $(p^1, p^2) = (0.55, 0.55)$ and $(p^1, p^2) = (0.45, 0.45)$. At these belief assignments types incorrectly agree that one of the types is more likely to receive high endowment. As the common belief is reflected in the bond price the believed-to-be-poor type turns out to be poor in fact. So, society’s welfare is low because discrepancy between agents’ individual welfare levels is large. The lowest welfare under the complete markets design is achieved at $(p^1, p^2) = (0.45, 0.55)$ and $(p^1, p^2) = (0.55, 0.45)$ (depicted by gray circles
in the figure). At these assignments beliefs are maximally heterogeneous. So, speculative motives are strong and survival forces occasionally drive each agent arbitrarily close to loosing all wealth. As a result consumption is very volatile and society’s welfare is low. We conclude that the bond-only design dominates the complete markets design:

\[ -2.132 = \min \mathcal{P} \sum_{i=1}^{I} \theta^i U^i_{p_0}(\mathcal{P}|CM) < \min \mathcal{P} \sum_{i=1}^{I} \theta^i U^i_{p_0}(\mathcal{P}|B) = -2.011. \]

Figure 9: Welfare in example 1: the bond-only (black) vs the complete markets (grey) design. Square point denotes the unconstrained maximum: \((p^1, p^2, W^*) = (0.5, 0.5, -2)\). Circle points denote belief assignments that attain the lowest welfare under the corresponding design.
Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 8\).

We now turn to the comparison of the complete markets economy with borrowing limits and the unrestricted complete markets design. We impose a borrowing limit of \(B = 1\) that equals two average annual incomes. This limit is very restrictive if compared with the natural borrowing limits. But it
Figure 10: Welfare in example 2: the complete markets with borrowing limits (black) vs the complete markets design (gray). Square point denotes the unconstrained maximum: \((p^1, p^2, W^*) = (0.5, 0.5, -2)\). Circles denote belief assignments that attain the lowest welfare under the corresponding design. Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 1\).

would not be binding in a competitive equilibrium with complete markets if both agents had correct beliefs. With diverse beliefs, on the other hand, any tight exogenous borrowing limit must be binding. Moreover, any borrowing limit is more restrictive when agents are pessimistic rather when they are optimistic.\(^{14}\) For this reason the worst belief assignment is \((0.45, 0.55)\) – when disagreement is maximal and both types are pessimistic. A tight borrowing limit restricts the maximal amount of wealth can be lost by any agent and bounds consumption away from zero. Yet, borrowing limits are activated only when wealth becomes unevenly distributed. When disagreement is small wealth is always close to being equally distributed and borrowing limits rarely bind. For this reason, the two market designs deliver similar welfare levels when disagreement is small. That is imposing even the tight borrowing

\(^{14}\)This is so because insurance and speculative motives align and prompt types to purchase more Arrow securities for the low income state. Cite Viktor.
limit $B = 1$ is nearly “harmless.” So, we conclude that the design with the borrowing limit $B = 1$ dominates the design with natural borrowing limits:

$$-2.132 = \min \mathcal{P} \sum_{i=1}^{l} \theta_i^1 U_{P^0}^i (\mathcal{P} | CMT) < \min \mathcal{P} \sum_{i=1}^{l} \theta_i^1 U_{P^0}^i (\mathcal{P} | CMB) = -2.028.$$ 

Finally, we analyze adding a transaction tax on each financial transaction as specified in (14). Imposing a transaction tax forces agents to take smaller positions than they otherwise would. This limits speculation but also restricts hedging possibilities. As above the new restriction is stronger when agents are pessimistic. So, the worst belief assignment under both designs is $(p^1, p^2) = (0.45, 0.55)$; see figure 11. Welfare is lowest in this case because 1) survival forces are at their peak potential and 2) agents’ pessimism prompts them to trade/hedge more actively making the transaction tax harming. Yet, imposing the transaction tax $\kappa = 0.05$ increases the society’s welfare:

$$-2.118 = \min \mathcal{P} \sum_{i=1}^{l} \theta_i^1 U_{P^0}^i (\mathcal{P} | CM) < \min \mathcal{P} \sum_{i=1}^{l} \theta_i^1 U_{P^0}^i (\mathcal{P} | CMB) = -2.049.$$ 

In these examples market weights are close to 0.5 as can be seen from figure 8. Because of this welfare comparisons are still driven by the least-advantaged type. Type weights is the only freedom that we leave to the modeler. But at least with the two choices of weights contemplated here we reached the same conclusion: all financial restrictions studied above are desirable. Even quantitatively results are comparable.

9 Liberating $P^0$

Our welfare criterion uses three min operators. However, so far we demonstrated the use of the criterion when the set of possible data generating processes $\mathcal{B}^0$ was a singleton. So, the outermost minimization was trivial. In this section we confront our designer with multiple data-generating processes: $|\mathcal{B}^0| > 1$. Notice, that our theoretical results hold for any $P^0$ and, hence, any $\mathcal{B}^0$. Our numerical examples also show that welfare varies more under the complete markets design than under the designs with financial restrictions. By introducing uncertainty about $P^0$, via expansion of $\mathcal{B}^0$, we expect welfare
Figure 11: Welfare in example 3: complete markets with borrowing limits and transaction tax (black) vs complete markets with borrowing limit (gray). Square point denotes the unconstrained maximum: \((p^1, p^2, W) = (0.5, 0.5, -2)\). Circle points denote belief assignments that attain the lowest welfare under the corresponding market design.

Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 8, \tau = 0.05\).

gains from financial restrictions to rise. These expectations are confirmed by the results reported in table 2.\(^{15}\)

Table 2 assumes that \((p^1, p^2) \in [0.45, 0.55]^2\). That is for each choice of \(p^0 \in [0.45, 0.50]\) we report \(\min_{P^1, P^2} \min_i W^i_{p^0}(c(P))\).\(^{16}\) Columns 2 through 5 present welfare under the unrestricted complete markets design (section 5.1), the bond economy (section 5.2), the complete markets with borrowing limits (section 6.4), and the complete markets with transaction tax (section 6.5) respectively. All the financial designs achieve the lowest welfare at \(p^0 = 0.45\). Welfare under the complete markets is \(W(CM) = -2.545\), the lowest among

\(^{15}\)We use the following notation: \(W_{p^0}(M) = \min_{P^1, P^2} \min_i U^i_{p^0}(c(P), M)\).

\(^{16}\)The results for \(p^0 \in [0.50, 0.55]\) are symmetric. So, both at \(p^0 = 0.55\) and at \(p^0 = 0.45\) we get \(W(CMB) = -2.171, W(CM) = -2.545\). Only the identity, type 1 or type 2, of the least well-off agent changes.
Table 2: Welfare level under different $P^0$: the designs with financial restrictions ($B,CMB,CMT$) vs the complete markets design (CM).

Parameters: $\beta = 0.96, \epsilon_l = 1/3, \epsilon_h = 2/3, p^0 = 0.5$.

<table>
<thead>
<tr>
<th>$P^0$</th>
<th>$W_{p^0}(CM)$</th>
<th>$W_{p^0}(B)$</th>
<th>$W_{p^0}(CMB)$</th>
<th>$W_{p^0}(CMT)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-2.545</td>
<td>-2.084</td>
<td>-2.121</td>
<td>-2.234</td>
</tr>
<tr>
<td>0.46</td>
<td>-2.439</td>
<td>-2.068</td>
<td>-2.113</td>
<td>-2.193</td>
</tr>
<tr>
<td>0.47</td>
<td>-2.347</td>
<td>-2.053</td>
<td>-2.106</td>
<td>-2.154</td>
</tr>
<tr>
<td>0.48</td>
<td>-2.267</td>
<td>-2.039</td>
<td>-2.098</td>
<td>-2.117</td>
</tr>
<tr>
<td>0.49</td>
<td>-2.195</td>
<td>-2.025</td>
<td>-2.090</td>
<td>-2.094</td>
</tr>
<tr>
<td>0.50</td>
<td>-2.139</td>
<td>-2.011</td>
<td>-2.083</td>
<td>-2.079</td>
</tr>
</tbody>
</table>

The best performing design with restricted financial markets is the bond-only economy that achieves welfare level $W(B) = -2.084$. It offers an improvement over the complete markets design equivalent to a permanent 22.1% increase in consumption. The design with borrowing limit $B = 1$ dominates the design with transaction tax $\tau = 0.05$: $W(CMB) = -2.121 > -2.234 = W(CMT)$. The former offers substantial improvement over the complete markets design, equivalent to a permanent 20.0% increase in consumption. But it underperforms relative to the bond-only design. The design with a transaction tax does not perform well when $p^0 \neq 0.5$. This happens because as $p^0$ diverges from 0.5 agents must take larger financial positions, that are costly due to a transaction tax, to hedge income fluctuations.\(^{17}\)

The worst case beliefs assignment for the complete markets design is $(p^0, p^1, p^2) = (0.45, 0.45, 0.55)$. It assigns correct beliefs to type-1 agents and maximally wrong beliefs to type-2 agents. The worst-case choice of beliefs maximizes strength of the survival forces.\(^{18}\) Type-2 agents have lowest wel-

\(^{17}\)To build intuition consider the case with correct and homogeneous beliefs. Recall that in the initial state, $z = 0$, both agents receive the same income 0.5. When $p^0 = 0.5$ agents trade to reallocate income across states. When $p^0 \neq 0.5$ agents get an additional motive to trade: to reallocate income across time. This motive appears because expected individual income is no longer 0.5 and agents want to borrow or lend against the future income. Because trading is costly agents end up with ‘suboptimal’ positions. See also derivations in section 6.

\(^{18}\)There is no more need to make the wealth effect operational by tilting everyone’s
fare in this economy. For the bond-only design the worst-case assignment of beliefs is $(p^0, p^1, p^2) = (0.45, 0.55, 0.535)$. Type-1 agents have the lowest welfare. Under this belief assignment type-1 agents wrongly believe that they are more likely to receive high endowment. So, they dissave and end up consuming less than type-2 agents. In addition, type-1 agents have less accurate beliefs guiding them to worse financial decisions. But, because of endogenous adjustment of the bond return and limited speculation opportunities type-1 agents loose wealth very slowly. This makes the bond-only economy a substantially more robust design than the complete markets. Under the complete markets with a borrowing limit the worst case assignment of beliefs is $(p^0, p^1, p^2) = (0.45, 0.515, 0.55)$. Type-2 agents have the lowest welfare. First, because their beliefs are less accurate. Second, because both types agree that type-2 agents are less likely to receive high endowment. This forces type-2 agents to stay close to a restrictive borrowing limit. However, unlike under the complete markets design, the strict borrowing limit $B = 1$ allows type-2 agents to re-build their financial wealth quickly. Under the complete markets with a transaction tax the worst case assignment of beliefs is $(p^0, p^1, p^2) = (0.45, 0.45, 0.55)$. Type-2 agents have the lowest welfare. This design is better than the complete markets because a transaction tax limits speculation. But a transaction tax also limits the speed of type-2 agent’s recovery once he runs into financial trouble. This makes the design with a transaction tax worse than the others.

The larger $B^0$ and $B$ the starker the welfare difference will be. Reasonable choices of $B^0$ and $B$ can be constructed using error detection probabilities as in Hansen and Sargent [2007].

9.1 Putting it to work

The great benefit of our criterion is that it can be applied “as is.” We demonstrate how to compute an optimal borrowing limit for our economy. Consider the complete markets financial design with borrowing limits. Before we imposed an exogenous borrowing limit $B = 1$. We now compute an optimal borrowing limit:

$$B^* = \arg \max_B \min_{p^1, p^2, p^0} \min_i W^i_{p^0}(CMB).$$

This approach allows forming a set of models that are reasonably hard to distinguish using a log-likelihood ratio test and a finite data sample.
In the example the optimal borrowing limit $B^*$ is 36% of an average annual income. In the economy with homogeneous beliefs agents would borrow 33% of an average annual income. So, the optimal borrowing limit is just over what is needed to hedge income fluctuations.

10 Learning

One could raise a question if financial market incompleteness would be justified when agents could learn from their experience. With learning allowed the complete markets design becomes more attractive, but it does not debase financial restrictions completely. We demonstrate this with an example. Consider the two-state two-agent economy from other examples. The truth is a first-order Markov process. But agents believe that it is an iid process. Agent $i$ starts with a Beta prior about the probability of state 1: $\hat{p}^i \sim B(n^i_0, m^i_0)$. Counters $n^i_0$ and $m^i_0$ represent prior number of realizations of state 1 and 2 correspondingly. Under this prior mean probability of state 1 is $n^i_0/(n^i_0 + m^i_0)$. Agents update their beliefs according to Bayes law that amounts to tracking the number of state 1 and 2 occurrences. We consider the following set of admissible prior distributions: $(n^i_0, m^i_0) \in \{(8.0, 12.0), (8.4, 11.6), (8.8, 11.2), \ldots, (12.0, 8.0)\}$. The implied set of admissible prior state-1 probabilities is $\{0.40, 0.42, 0.44, \ldots, 0.60\}$. Holding the truth fixed at $p^0_1 = 0.50$, we compute society’s welfare under all possible assignments of priors. We do computations for two financial market designs: the complete financial markets and the complete financial markets with borrowing limits. The two welfare surfaces are presented in figure 12. It resembles closely figure 6 that compares the same designs but does not allow learning. With the restrictive borrowing limit speculation is limited and the worst case corresponds to nearly homogeneous priors: $B(11.6, 8.4)$ and $B(12, 8)$ for type-1 and type-2 agents respectively. Under the first prior assignment type-2 agent has lower welfare because both types initially agree that type-2 agents are unlikely to receive high income. Under the complete markets design the worst case assignment of beliefs is $B(10.4, 9.6)$ and

\begin{footnotesize}
\begin{itemize}
\item[20]This is not the natural borrowing limit but an equilibrium borrowing amount.
\item[21]We rule out learning from endogenous outcomes, e.g. prices. We believe that in realistic environments, in which the truth may not even be in the set of admissible beliefs, it is impossible to deduce any meaningful information from prices.
\item[22]Figure 12 plots interpolated data to provide the same resolution as other figures.
\end{itemize}
\end{footnotesize}
$B(12, 8)$ for type-1 and type-2 agents respectively. A type-2 agent is impoverished because his prior is less accurate in addition to being believed unlikely to receive high income.

In this setting agents’ beliefs converge to the truth. So, agents agree eventually and enjoy stable consumption. But consumption in distant periods contributes little to agents’ welfare. If there is substantial disagreement initially there will also be substantial speculation in the short and medium run. And if learning were slow, as for example in Cogley et al. [2012] where agents disagree about rare events, then speculation would persist and even patient agents would have low welfare.

Figure 12: Welfare in example 2 with learning: complete markets with borrowing limits (black) vs complete markets (gray). Square point denotes the unconstrained maximum: $(p^1, p^2, W) = (0.5, 0.5, -2)$. Circle points denote belief assignments that attain the lowest welfare under the corresponding market design.
Parameters: $\beta = 0.96, e_l = 1/3, e_h = 2/3, p^9 = 0.5, B = 1$. 
11 On choice of preference specification

We made two crucial assumptions about individual preferences. The first is that the preferences are time separable and the second is that the period utility function is unbounded below. Neither is crucial for our analysis and we provide our arguments below.

Suppose that individual preferences have a recursive utility representation as in Epstein and Zin [1989]. When markets are complete and agents have diverse beliefs some agent types will be driven out of financial markets. The difference is, as Borovicka [2012] shows, that it may not be the agent with the most accurate beliefs who survives as in Blume and Easley [2006]. But as long as there are agents that could be driven out of financial markets there is need for financial regulation. Our arguments are also more compelling in this case because speculation may impoverish agents with more accurate beliefs.

When period utility function is bounded from below then we believe that survival forces could be stronger because potential financial losses have lower utility cost. We demonstrate this by changing the period utility specification to $u(c) = \sqrt{c}$. Figure 17 in appendix C plots welfare surfaces for the complete markets design and the complete markets with an exogenous borrowing limit design. It shows that the welfare effect of imposing the borrowing limit $B = 1$ is less significant than with $u(c) = -1/c$ and it is equivalent to a 5.59% permanent increase in consumption. But the set of beliefs for which the complete markets is a preferred financial design is much smaller than under $u(c) = -1/c$. To paraphrase, survival forces are stronger and agents can loose financial wealth more quickly, but the welfare effect of loosing wealth is not as significant.

Finally, we would like to comment on using the minimum welfare in the society. Consider instead any fixed weighting of society members:

$$\min_{P^0 \in B^0} \min_{P \in B^l} \sum_{i=1}^{I} \theta^i E_{P^0} \left[ \sum_{t=0}^{\infty} \beta^t u(c_i^t(\sigma|\mathcal{P},\mathcal{M})) \right].$$

Because the period utility function is concave the society’s welfare is largely determined by the lowest individual welfare level. So, using the new criterion would yield similar predictions.$^{24}$ Figure 16 in appendix B illustrates the new

$^{23}$Welfare levels under the two financial designs are respectively 1.3033 and 1.3762. Welfare level in the economy in which agents hold correct beliefs is 1.4142.

$^{24}$In fact our theoretical results must hold true for the ‘average’ welfare criterion.
criterion with \( \theta^1 = \theta^2 = 0.5 \). As expected, the results are qualitatively very similar to the case with the original welfare criterion. Imposing a borrowing limit \( B = 1 \) leads to a welfare improvement that is equivalent to a 5.12% permanent increase in consumption.25

12 Concluding remarks

In this work we propose a framework and a welfare criterion for evaluation of different financial market designs. Our setting is an endowment economy in which agents may hold heterogeneous beliefs. We imagine a designer that chooses a financial market design before beliefs and the true data generating process are assigned. Our designer chooses a market structure that serves to the maximum benefit of the least advantaged agent in the society. This level of abstraction allows us to achieve a maximally objective choice. In fact, according to Rawls [1971] the approach adopted here is the only way to achieve objectiveness.

We use our criterion to study a simple economy. We highlight welfare-reducing speculation and welfare-improving insurance possibilities. The strength of each depends on a particular design of financial markets. We find that financial market designs with simple restrictions like borrowing limits and transaction costs offer substantial welfare gains relative to the complete financial markets benchmark. In our examples gains can be as large as those stemming from a 6% permanent increase in consumption.

Our theoretical results state that the financial autarky design dominates the complete markets design as long as disagreement can be large or agents are sufficiently patient. Unfortunately, we have no theoretical results for economies with intermediate levels of financial market incompleteness.

Finally, we believe that the biggest criticism of our analysis is the absence of incentive effects. That is restrictions imposed on the financial markets have no affect on the set of feasible allocations. This is arguably the most profitable deviation in this line of work.

25The minimal welfare levels in the two economies are respectively -2.1320 and -2.0281.
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### A Complete markets design

In this section we explain the shape of the welfare surface under the complete markets design. Two forces are key to understanding this surface. The first is the survival force: the type of agent with the least accurate beliefs is driven out of the market and is likely to have the lowest welfare. The second is the wealth effect: an equilibrium price system is affected by the configuration of beliefs and this may present an advantage to one of the types.\(^{26}\)

Figure 13 reproduces the welfare surface shown in figure 4. Along arc AOB both types are either optimistic or both pessimistic. The wealth effects for each type offset each other. So, the welfare is decreasing as we move away from point O because agents disagree more on individual states and accept more volatile consumption. When we perturb beliefs slightly away from the arc welfare drops. This happens because of the wealth effect. Independently of the direction in which beliefs are perturbed, one type’s wealth will be affected negatively and this reduces his and the society’s welfare.

Along arc CD both types are close to agreement but \(\text{prob}^1(\sigma_t = 1) > \text{prob}^2(\sigma_t = 1)\). Consider the closer half of arc CD where \(\text{prob}^2(\sigma_t = 1) \geq 0.5\). Then, a type-1 agent is optimistic and a type-2 agent is pessimistic. This configuration of beliefs is advantageous to a type-1 agent. (See also our two period example in the text.) But a type-1 agent also has less accurate beliefs. So, he is affected adversely by survival forces. The latter partially offsets the wealth effect and creates a ridge along arc CD.\(^{27}\)

\(^{26}\)When beliefs are equally accurate, the direction can be determined by looking at the date-0 consumption level. If the wealth effect impacts both types equally then \(c^*_0 = 0.5\).

\(^{27}\)Along the more distant half of arc CD the roles of the two types reverse.
Figure 13: Welfare in example 1 under the complete markets design (gray). Square point denotes the unconstrained maximum: \((p^1, p^2, W) = (0.5, 0.5, -2)\). Circle points denote belief assignments that attain the lowest welfare.

Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5\).

### A.1 Wealth effect

It is instructive to study a simple two-period economy. This example demonstrates that an agent with less accurate beliefs can secure a higher objective welfare. The key to this result is a wealth effect.

A period utility function is \(u(c) = \log(c)\) and future utility is not discounted. The state in period 0 is known, and there are two possible state realizations in period 1. Endowments for the two types are \((0.5, 0.5)\) in period 0, and in period 1 they are \((1, 0)\) if the state is 1 and \((0, 1)\) if the state is 2. Under the true probability distribution both states are equally likely. A type-1 agent’s beliefs coincide with the truth. But a type-2 agent believes that \(\text{prob}(s = 1) = 0.5(1 - \Delta) \neq 0.5\). Depending on whether \(\Delta > 0\) or \(\Delta < 0\) a type-2 agent is optimistic or pessimistic.

If both types had correct beliefs, in a competitive equilibrium allocation
with complete markets every agent would consume 0.5 in every period and state.

\[ c_2^0 = \frac{1}{2 - \Delta}, \quad c_2^1(s = 1) = \frac{1 + 2\Delta}{2 + \Delta}, \quad c_2^1(s = 2) = \frac{1 - 2\Delta}{2 - 3\Delta}. \quad (17) \]

There are two aspects of this equilibrium that are important. First, consumption of agent 2 is decreasing on average for all \( \Delta \neq 0 \):

\[ E[c_2^1] = c_0^2 \frac{4 - 4\Delta^2 c_0^2}{4 - \Delta^2 (c_0^2)^2} < c_0^2. \quad (18) \]

That is the agent with incorrect beliefs is being "driven out from the market." Second, if a type-2 agent is optimistic \( (p < 0) \) then his consumption in period 0 is higher than 0.5. Lastly, the agent with incorrect beliefs may have higher objective welfare:

\[ \frac{dW^2(\Delta)}{d\Delta} \bigg|_{p=0} = 1 \neq 0, \quad (19) \]

where \( W^2(\Delta) \equiv ln(c_0^2) + 0.5ln(c_2^1(s = 1)) + 0.5ln(c_2^1(s = 2)) \). That is agent 2 can be better off being an optimist. But \( \lim_{p \to 0.5} W^2(p) = -\infty \). Figure 14 plots welfare of the two types of agent. The horizontal dotted line denotes the
welfare level in the economy in which beliefs of each agent coincide with the truth. A type-2 agent benefits from being optimistic because of his impact on the equilibrium price system. Optimism increases the relative price of goods delivered in state \( s = 2 \). This is the wealth effect.

### A.2 Symmetric beliefs

In this section we show how to construct a symmetric beliefs assignment. We continue to assume that endowments are ex-ante identical. This means that for every \( t \) and \( \sigma \) there exist \( \sigma' \) such that \((e_1^t(\sigma), e_2^t(\sigma)) = (e_2^t(\sigma'), e_1^t(\sigma'))\). Symmetric beliefs have the property that for every \( t \) and \( \sigma \) there exists \( \sigma' \) with \((P_1^t(\sigma), P_2^t(\sigma)) = (P_2^t(\sigma'), P_1^t(\sigma'))\). That is identities of the two agents are fully interchangeable.

We provide a simple example now. Suppose that there are two periods and two paths: \( \sigma, \sigma' \). Endowments are specified as follows: \( e_0(\sigma) = e_0(\sigma') = (1, 1), e_1(\sigma) = (0, 1), e_1(\sigma') = (1, 0) \). We will now assign beliefs to the two paths. For any \( p \in [0, 1] \) assign \( \text{prob}^1(\sigma) = p, \text{prob}^1(\sigma') = 1 - p, \text{prob}^2(\sigma) = 1 - p, \text{prob}^2(\sigma') = p \). We say that this system of beliefs is symmetric. If financial markets were complete different endowment streams would be valued equally. To put it differently, a competitive equilibrium allocation would be a solution to a Pareto problem with equal weighing of agents.

### B Alternative welfare criterion

In this section we explore an alternative welfare criterion that uses implied Pareto weights from a competitive equilibrium with homogeneous beliefs.

**Definition.** Let \( \mathcal{B} \) be a set of admissible beliefs and let \( \mathcal{P} = (P^1, \ldots, P^I) \in \mathcal{B}^I \) denote a belief assignment. Let \( P^0 \in \mathcal{B}^0 \) be a data generating process, where \( \mathcal{B}^0 \) is a set of admissible data generating processes. Let \( (\theta^1, \ldots, \theta^I) \in \Delta^{I-1} \) be a set of welfare weights. Let \( c(\mathcal{P} | \mathcal{M}) \) be a competitive equilibrium allocation under a financial market structure \( \mathcal{M} \) and a belief assignment \( \mathcal{P} \). Then the social welfare function is:

\[
\min_{P^0 \in \mathcal{B}^0} \min_{\mathcal{P} \in \mathcal{B}^I} \left[ \sum_{i=1}^{I} \theta^i W^i_{p_0}(c^i(\mathcal{P} | \mathcal{M})) \right].
\]

(20)

Of course, we will obtain different answers for different choices of weights.
Is there an “objective” choice of weights? One candidate is the set of Pareto weights that corresponds to the competitive equilibrium outcome under homogeneous beliefs. This set is the solution\(^{28}\) to the following system of equation:

\[
\theta^i = \frac{[u'(c_0^i(P|CM))]^{-1}}{\sum_{h=1}^{I}[u'(c_0^h(P|CM))]^{-1}}
\]

The gray surface in figure 15 represents the “alternative” welfare criterion. The black surface corresponds to our “Rawlsian” criterion. As expected the “Rawlsian” welfare surface is never larger than the “alternative”. Also the two surfaces coincide where \(p^2 = 1 - p^1\) because then agents are symmetric and \(\theta^1 = \theta^2 = 0.5\). The lowest welfare levels are -2.132 and -2.139 for the “alternative” and the “Rawlsian” criteria. These are achieved at belief assignments where disagreement is maximal or close to maximal. So, it seems like the two criteria would lead to similar decisions. The advantage of our criterion is that we release our designer of computing the set of welfare weights. And if one agreed with our choice of weights, one should also adopt our criterion as it would lead to a similar, if not the same, ranking of financial structures.

Finally, one could use equal weights for each type. So, the task of a designer would be to maximize the worst case average welfare. Under the assumption that utilities are unbounded below, this criterion would produce results very similar to ours. The reason is that the average welfare would largely be driven by the least advantaged type. Figure 16 plots welfare surfaces for the complete markets and the complete markets with the borrowing limit \(B = 1\) designs.

C Utility that is bounded below

\(^{28}\)It is unique given our assumptions on preferences.
Figure 15: Welfare in example 1 under the complete markets design: “alternative” (gray) vs “Rawlsian” criterion. Circle points denote belief assignments that attain the lowest welfare under the corresponding design.
Parameters: $\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5$. 
Figure 16: Welfare in example 2: the complete markets with borrowing limits (black) vs the complete markets design (gray). Square point denotes the unconstrained maximum: \((p^1, p^2, W^*) = (0.5, 0.5, -2)\). Circles denote belief assignments that attain the lowest welfare under the corresponding design. Parameters: \(\beta = 0.96, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 1\).
Figure 17: Welfare in example 2: the complete markets with borrowing limits (black) vs the complete markets design (gray). Square point denotes the unconstrained maximum: \((p^1, p^2, W^*) = (0.5, 0.5, -2)\). Circles denote belief assignments that attain the lowest welfare under the corresponding design. Parameters: \(\beta = 0.96, u(c) = \sqrt{c}, e_l = 1/3, e_h = 2/3, p^0 = 0.5, B = 1\).